More Green's Theorem and Simple Connectedness

Saturday, April 17, 2021 9:01 AM

Recall in $\mathbb{R}^n$

Definition $\vec{F}$ is conservative in a region R if $\exists F$ a scalar fcn on R s.t. $\vec{F} = \nabla F$, then "F is a potential for $\vec{F}$ on R"

Note If R is connected, then any 2 potentials for the same $\vec{F}$ differ by a constant.

Thm For $\vec{F}$ on R (connected):

conservative $\iff$ path independent $\iff$ $\int_C \vec{F}\, d\vec{s} = 0$ for C closed

Define for a vector field $\vec{F}$ in $\mathbb{R}^2$, the curl of $\vec{F}$ is the scalar fcn $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ for $\vec{F} = P\vec{i} + Q\vec{j}$

Proposition If $\vec{F}$ is conservative on R, then curl $\vec{F} = 0$

Proof Let F be a potential for $\vec{F}$ (on R)

Then

$$P = \vec{F}\cdot\vec{i} = \frac{\partial F}{\partial x}$$

$$Q = \vec{F}\cdot\vec{j} = \frac{\partial F}{\partial y}$$

$$\Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$

These are equal $\left(\frac{\partial}{\partial x \partial y} = \frac{\partial}{\partial y \partial x}\right)$

$$\Rightarrow \text{curl } \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

■ QED

Next goal for certain regions R, curl $\vec{F} = 0 \Rightarrow \vec{F}$ is conservative

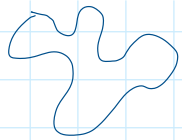
Jordan Curve Thm

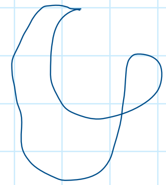
Definition A simple closed curve C in $R \subseteq \mathbb{R}^2$ is a curve given by the parameterization $(x(t), y(t))$ for $a \le t \le b$ s.t.:

1. $(x(a), y(a)) = (x(b), y(b))$ ← closed
2. If $a \le t_1 < t_2 \le b$ then $(x(t_1), y(t_1)) \ne (x(t_2), y(t_2))$ unless $t_1 = a$ & $t_2 = b$ ("doesn't cross itself")

e.g.

Simple closed Not simple closed





Thm (Jordan Curve Thm)

If C is a simple closed curve in $\mathbb{R}^2$, then C divides $\mathbb{R}^2$ into 2 regions:

- one bounded by $\text{int}(C)$
- one unbounded $(\text{ext}(C))$

In fact, $\forall P \in \mathbb{R}^2$ either $P \in \text{ext}(C)$, $P \in C$ or $P \in \text{int}(C)$

eg $C = \left\{ \vec{r} \in \mathbb{R}^2 \;\middle|\; \|\vec{r} - \vec{r}_0\| = r \right\}$

$= C_{r_0}(r)$

= circle of radius r and center $\vec{r}_0$

(simple closed if $r > 0$)

$D = D_{\vec{r}_0}(r) = \left\{ \vec{r} \in \mathbb{R}^2 \;\middle|\; \|\vec{r} - \vec{r}_0\| < r \right\}$

"open disc"

then $D = \text{int}(C)$

Remarks

1. C is a point iff $a = b$ (in parameterization) iff $\text{int}(C) = \emptyset$
2. C = boundary $(\text{int}(C))$ = "$\partial(\text{int}(C))$"

Thm (Green's Thm)

Suppose $\vec{F} = P\vec{i} + Q\vec{j}$ is defined & diff'able on some open domain $D \subseteq \mathbb{R}^2$

(technical point: need $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial y}$ all continuous)

and C is a simple closed curve in D s.t. $\text{int}(C) \subseteq D$

then
$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$$
$$= \int_{\text{int}(C)} \text{curl}\,\vec{F}\; dx\,dy$$

Definition Let R be an open region in $\mathbb{R}^2$

We say R is simply connected if $\forall$ simple closed C contained in R, we also have $\text{int}(C) \subseteq R$

eg

- disc $D_{\vec{r}_0}(r)$

non-eg

- $\mathbb{R}^2 \setminus \{(0,0)\}$

eg

- disc $D_{\vec{r}_0}(r)$
- interior of a triangle
- interior of a convex polygon
- a half plane

  eg:

  $\{(x,y) \mid x>0\}$

  $\{(x,y) \mid y<2\}$

  $\{(x,y) \mid ax+by<c\}$
- $\{P \in \mathbb{R}^2 \mid a < \theta < b\}$
- quadrant

non-eg

- $\mathbb{R}^2 \setminus \{(0,0)\}$
- $\mathbb{R}^2 \setminus \{(7,-\sqrt{2})\}$
- $\mathbb{R}^2 \setminus \{(0,0),(1,2)\}$
- $\mathbb{R}^2 \setminus S$, for S a nonempty finite set of points
- $\mathbb{R}^2 \setminus S$, S any bounded nonempty closed subset

  eg: $S = \overline{D_{\vec{r}_0}(1)}$ ← a closed disc of radius 1
- any nonempty disc with a finite nonzero number of points removed

  eg: $D_{(3,4)}(2) \setminus \{(3,5),(4,3)\}$

Prop If S is a bounded nonempty closed subset of $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus S$ is not simply connected.

Proof Since S is bounded, then

$S \subseteq D_{(0,0)}(r)$ for sufficiently large r ($r \gg 0$)

$\Rightarrow S \cap C_{(0,0)}(r) = \emptyset$

$\Rightarrow C_{(0,0)}(r) \subseteq \mathbb{R}^2 \setminus S$

but since S is nonempty, can find some $P \in S$

$\Rightarrow P \in D_{(0,0)}(r) = \text{int}(C_{(0,0)}(r))$

$\Rightarrow \text{int}(C_{(0,0)}(r)) \not\subseteq \mathbb{R}^2 \setminus S$

$\mathbb{R}^2 \setminus S$ is NOT simply connected

Thm Suppose $\vec{F}$ is a vector field on a simply-connected region R and $\text{curl}(\vec{F}) = 0$. Then $\vec{F}$ is conservative on R.

Proof We will show $\vec{F}$ is conservative by showing $\int_C \vec{F}\cdot d\vec{r} = 0$

$\forall$ closed curve $C \subseteq R$.

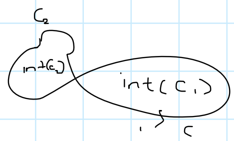
If C is simple closed then

$$\int_C \vec{F}\cdot d\vec{r} = \int_{\text{int}(C)} \text{curl}(\vec{F})\,dx\,dy$$

$$= \int_{\text{int}(C)} 0\,dx\,dy$$

$= 0$

In general, suppose C crosses itself, eg:



divide C into 2 simple closed curves C_1 & C_2

Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot dr$$

because $C = C_1 - C_2$

minus bc CCW around C_1 goes CW around C_2

Now

Since the integrals over closed simple curves are 0, so is $\vec{F} \cdot d\vec{r}$

QED

eg Let $\vec{F}(x, y) = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$

Notice

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$\Rightarrow \text{curl}(\vec{F}) = 0$

(except at $(x, y) = (0, 0)$ where $\vec{F}$ is undefined)

$\vec{F}$ is a vector field on $\mathbb{R}^2 \setminus \{(0,0)\}$ but $\mathbb{R}^2 \setminus \{(0,0)\}$ isn't simply connected